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Peano differentiable extensions in o-minimal structures

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Abstract. Peano differentiability generalizes ordinary differentiability to higher order. There are two ways to define Peano differentiability for functions defined on non-open sets. For both concepts, we investigate the question under which conditions a function defined on a closed set can be extended to a Peano differentiable function on the ambient space if the sets and functions are definable in an o-minimal structure expanding a real closed field.

Key words. o-minimal structure, Peano derivative, extensions.

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1 Introduction

1.1. The usual notion of differentiability, which we also call Fréchet differentiability, generalizes to higher order in two ways. Let m be a positive integer.

Firstly, we say that a function f is m *times Fréchet differentiable* if f is Fréchet differentiable, and all partial derivatives of f are $m-1$ times Fréchet differentiable.

Secondly, we stipulate that the function satisfies the Taylor formula of order m at every point, cf. [18]. Let us make this notion, called Peano differentiability, precise.

Suppose U is an open subset of $\mathbb{R}^n$. We say that the function $f : U \to \mathbb{R}$ is m *times Peano differentiable* at $u \in U$, if there exists an *approximation polynomial* p of degree m with $p(0) = 0$, such that

$$f(x) - f(u) = p(x-u) + o(\|x-u\|^m) \text{ as } x \to u.$$

The function f is called m *times Peano differentiable* if f is m times Peano differentiable at every point of the domain.

Every m times continuously differentiable function is m times Fréchet differentiable, and every m times Fréchet differentiable function is m times Peano differentiable. The approximation polynomial at some point is uniquely determined by the function, so that

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for an m times continuously differentiable function, both the approximation polynomial and the Taylor polynomial coincide at any point of the domain. For details about the differences of these differentiability concepts, we refer the reader to [13, Example 2.5].

If the domain is closed, we can also define Peano differentiability. Contrary to open domains, the approximation polynomials are not necessarily uniquely determined by f, and also Taylor's theorem does not apply anymore, cf. [5]. Therefore, in analogy to continuous differentiability in the sense of Whitney, cf. [27], one either fixes for every point u a certain approximation polynomial; or, one is only interested in the existence of approximation polynomials. In the latter case, we say that f is *weakly m times Peano differentiable*. In both cases it is natural to ask whether or not such a function is the restriction of an m times Peano differentiable function defined on the ambient space.

1.2. Originating in model theory, o-minimal structures are objects of enormous geometrical potential. Let R be a real closed field. A *semialgebraic set* is a Boolean combination of sets of the form $\{x \in R^n : p(x) \geq 0\}$, where p is a polynomial in n variables with coefficients of R.

An o-minimal expansion $\mathcal{M}$ of R is a sequence of sets $(M_n)_{n\in\mathbb{N}}$ which satisfies the following axioms.

(S1) Each M_n is a Boolean algebra of subsets of R^n, which are called *definable*.
(S2) Every semialgebraic subset of R^n is definable.
(S3) If A and B are definable, then $A \times B$ is definable.
(S4) Projections of definable sets are definable.
(O) The definable subsets of R precisely the finite unions of intervals and points.

A function f is called *definable* if the graph $\Gamma(f)$ is definable.

A sound introduction to o-minimality is the book [21], and [23] is an excellent survey from the geometrical perspective.

The collection of all semialgebraic sets forms an o-minimal expansion of R, cf. [3, Chapter 2]. On the real field, the globally subanalytic sets form the o-minimal structure $\mathbb{R}_{\text{an}}$. In this structure, all analytic functions restricted to compact cubes are definable. The real exponential field $\mathbb{R}_{\text{exp}}$, which consists of the smallest collection $(M_n)_{n\in\mathbb{N}}$ satisfying axioms (S1)–(S4) such that the graph of the entire exponential function is definable, forms also an o-minimal structure, cf. [28, p. 398]. The generation of o-minimal structures is an active research topic, and recent examples of o-minimal structures are constructed in [22, 24, 25, 16, 19].

1.3. For each natural number n we endow R^n with the Euclidean topology and the Euclidean norm $\|\cdot\|$. For a multi-index $\alpha \in \mathbb{N}^n$, let $\alpha! := \alpha_1! \dots \alpha_n!$ and $|\alpha| := \alpha_1 + \dots + \alpha_n$, and, if $x = (x_1, \dots, x_n)$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let X be a subset of R^n. A function $f = f_{(0,\dots,0)} : X \to R$ together with the functions $f_\alpha : X \to R$, where $1 \leq |\alpha| \leq m$, is called *m times Peano differentiable* if for all $x \in X$

$$f(y) - f(x) = \sum_{1\leq|\alpha|\leq m} \frac{f_\alpha(x)}{\alpha!}(y-x)^\alpha + o(\|y-x\|^m) \text{ as } y \to x.$$

We also say that $(f_\alpha)_{|\alpha|\le m}$ is m times Peano differentiable relative to X, and simply prefix the word *definable* if all functions f_α are definable. The functions f_α are called *Peano derivatives*. If f is a definable m times continuously differentiable function defined on an open set, then, by Taylor's theorem, the functions f_α coincide with the usual α-th derivative $D_\alpha f$.

If for each $|\alpha| \le m-1$, the function $(f_{\alpha+\beta})_{|\beta|\le m-|\alpha|}$ is additionally $m-|\alpha|$ times Peano differentiable, we say that f is m times Fréchet differentiable.

Moreover, we abbreviate m *times Peano, Fréchet and continuously differentiable* by the symbols $\mathscr{P}^m$, $\mathscr{F}^m$ and $\mathscr{C}^m$, respectively.

In this paper, we describe functions defined on closed sets, which can be extended as m times Peano differentiable functions, under the additional assumption that the sets and functions are definable in an o-minimal expansion of a real closed field.

1.4 Results. In general, let $m \ge 2$. The extension problem for Peano differentiable functions defined on closed subsets of $\mathbb{R}$ has been soundly studied in various papers, cf. [4, 17, 26]. The most general result on functions extendable as m times Peano differentiable functions is given by the following theorem, cf. [9], generalizing the corresponding result in [1] for Fréchet differentiable functions, and the one-dimensional case for Peano differentiable functions, cf. [8]. Note that a function f is called a *Baire*-1 function, if f is the pointwise limit of a sequence of continuous functions.

Theorem 1.1. *Let $X \subset \mathbb{R}^n$ be closed and let $f_{(0,\dots,0)} : X \to \mathbb{R}$ together with the functions $f_\alpha : X \to \mathbb{R}$, where $1 \le |\alpha| \le m$, be m times Fréchet differentiable, such that each f_α is a Baire-1 function for $|\alpha| = m$. Then there is a $\mathscr{P}^m$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F_\alpha = f_\alpha$ on X for all $|\alpha| \le m$.*

In the present paper, we study an o-minimal version of Theorem 1.1. That is, we assume that the sets and functions are definable in an o-minimal expansion of a real closed field.

We fix a real closed field R and an o-minimal expansion $\mathcal{M}$ of R. The main result is the following theorem, which generalizes [11, Theorem 1.3] for definable Fréchet derivatives.

Theorem 1.2.[1] *Let $A \subset R^n$ be a closed definable set. Let $(f_\alpha)_{|\alpha|\le m} : A \to R$ be definably m times Peano differentiable relative to A. If there exists a finite partition of A into definable sets $A_1, \dots, A_r$ such that for every $i = 1, \dots, r$*

$$(f_\alpha)_{|\alpha|\le m} \text{ is } m \text{ times Fréchet differentiable in } A_i, \tag{$*$}$$

then there is a definable m times Peano differentiable function $F : R^n \to R$ with

$$F_\alpha = f_\alpha \text{ on } A \text{ for all } |\alpha| \le m. \tag{1.1}$$

[1]This theorem is from the PhD thesis of the author.

For $n = 2$, condition $(*)$ is even necessary. We point out that the sets A_i need not to be closed. Moreover, the Baire-1 property is missing in the o-minimal case, as by [11, Theorem 1.5] every definable function satisfies the stronger definable Baire-1 property. That is, every definable function is the pointwise limit of a definable family of continuous functions.

We also study the problem of gluing Peano differentiable functions with closed domains together. This requires the following stronger equality concept.

Let $f, g : R^n \to R$ be functions, and let A be a subset of R^n. We say that f and g are $\mathscr{P}^m$ equal in A if for all $a \in A$

$$f(x) - g(x) \text{ is } o(\|x-a\|^m) \text{ as } x \to a.$$

If A is an open set, then this is equivalent to the usual equality of functions. For the following theorem, we also allow $m = 1$.

Theorem 1.3. *Let $A_1, \dots, A_r \subset R^n$ be definable closed sets, and let $f_1, \dots, f_r : R^n \to R$ be m times Peano differentiable functions, such that for $1 \le i < j \le r$*

$$f_i \text{ and } f_j \text{ are } \mathscr{P}^m \text{ equal in } A_i \cap A_j.$$

Then there exists an m times Peano differentiable function $F : R^n \to R$ such that for all $j = 1, \dots, r$

$$F \text{ and } f_j \text{ are } \mathscr{P}^m \text{ equal in } A_j.$$

If the functions $f_1, \dots, f_r$ are definable, we may choose F to be definable.

This gluing property does not apply to continuously differentiable functions, see [15, Theorem 5.5f] or [20, Proposition 4.7f]. We do not know whether there exist corresponding studies for Peano differentiable functions in classical analysis.

Last we study the extension problem for weakly Peano differentiable functions for functions defined on definable subsets of dimension 1, definable $\mathscr{P}^m$ manifolds, and on closed definable subsets of R^2. For subsets of R^2 we can give a complete answer in form of the following theorem.

Theorem 1.4. *Let $A \subset R^2$ be a closed definable subset. Let $f : A \to R$ be weakly m times Peano differentiable. Then there is an m times Peano differentiable function $F : R^2 \to R$ such that $F = f$ on A. If f is definable, we may choose F to be definable.*

The previous theorem holds in arbitrary dimension and for definable Fréchet differentiable functions ($m = 1$), cf. [11, Theorem 1.4].

There are 5 sections subsequent to the introduction. In Section 2 we discuss examples concerning the extendability of definable Peano derivatives. In Section 3 and 4 we prove Theorem 1.2 and Theorem 1.3, respectively. In Section 5 we discuss several special cases for the extendability of weakly Peano differentiable functions, and we prove Theorem 1.4. In Section 6 we formulate open questions about extending definable Peano derivatives.

2 Examples

For a subset S of R^n let $\chi_S : R^n \to R$ denote the characteristic function of S.

Next we give examples that show that, in general, we cannot extend m times Fréchet differentiable functions defined on closed sets to R^n to R^n to m times Fréchet differentiable functions.

Example 2.1. Consider the definable sets $B := \{(x, y) : x \geq 0 \wedge y \geq x^{m+1}\}$ and $C := \{(x, y) : x \leq 0 \vee y \leq 0\}$. Then $A := B \cup C$ is definable and closed. Define $(f_\alpha)_{|\alpha|\leq m} : A \to R$ by

$$f_\alpha(x, y) := \begin{cases} \frac{(m+1)!}{(m+1-\alpha_1)!} x^{m+1-\alpha_1} \chi_B(x, y), & \text{if } \alpha_2 = 0, \\ 0, & \text{if } \alpha_2 > 0. \end{cases}$$

Then $(f_\alpha)_{|\alpha|\leq m}$ is m times Fréchet differentiable relative to A, but there is not even a definable $\mathscr{C}^1$ function $F : R^2 \to R$ with $F = f_{(0,0)}$ on A.

Proof. Let $F : R^2 \to R$ be a definable Fréchet differentiable function that coincides with $f_{(0,0)}$ in A. For $x \geq 0$ we note that

$$F(x, x^{m+1}) - F(x, 0) = f(x, x^{m+1}) - f(x, 0) = x^{m+1}.$$

For $x > 0$, consider the map

$$y \mapsto F(x, y).$$

As F is Fréchet differentiable, we obtain by the Mean Value Theorem a number y_x between 0 and x^{m+1} such that

$$F(x, x^{m+1}) - F(x, 0) = \frac{\partial F}{\partial y}(x, y_x)(x^{m+1} - 0).$$

Therefore,

$$\frac{\partial F}{\partial y}(x, y_x) = \frac{1}{x},$$

which implies that the partial derivative of F with respect to y is not bounded at the origin, and thus it is not continuous. $\square$

Remark 2.2. If we take $R = \mathbb{R}$ in the previous example, we may conclude that there is not even a (non-definable) $\mathscr{C}^1$ function extending $f_{(0,0)}$ to $\mathbb{R}^2$.

Remark 2.3. Example 2.1 provides us with explicit functions that show that gluing of m times Fréchet differentiable functions ($m \geq 2$) without further conditions on the underlying sets does not preserve the differentiability class. The function f restricted to both B and C can be considered as restrictions of polynomials to B and C, respectively, and they are $\mathscr{P}^m$ equal in $B \cap C$. But there is no m times Fréchet differentiable function $F : R^2 \to R$ satisfying $F = f$ on B and $F = f$ on C.

The next example shows that the condition $(*)$ of Theorem 1.2 cannot be dropped without substitution. We recall the following fact. Let $f : U \to R$ be a definable $\mathscr{P}^m$ function, where U is an open subset of R^n. Then the set V, which consists of the points at which f is not $\mathscr{C}^m$ smooth, is definable, and

$$\dim(V) \leq n-2, \tag{2.1}$$

cf. [13]. In particular, if $n = 2$, then V is finite.

Example 2.4. Let $A := \{0\} \times R$. Let $(f_\alpha)_{|\alpha|\leq 2} : A \to R$ be defined by

$$f_\alpha(x,y) := \begin{cases} 0, & \text{if } \alpha \neq (1,1), \\ 1, & \text{if } \alpha = (1,1). \end{cases}$$

Then $(f_\alpha)_{|\alpha|\leq 2}$ is m times Peano differentiable relative to A, but there is no definable $\mathscr{P}^2$ function $F : R^2 \to R$ with

$$F_\alpha = f_\alpha \text{ on } A \text{ for all } |\alpha| \leq 2.$$

Proof. We assume that f is the restriction of a definable 2 times Peano differentiable function $F : R^2 \to R$. In this case, the set of points at which F is not $\mathscr{C}^2$ smooth is finite. In particular, the function $f_{(1,0)}$ is continuously differentiable with respect to the second variable outside of a finite set. This implies that

$$\frac{\partial f_{(1,0)}}{\partial y}(0,y) = f_{(1,1)}(0,y) = 1$$

except for finitely many $y \in R$. Obviously, this is not the case. $\square$

3 Proof of Theorem 1.2

3.1 Preliminary lemmas. In the sequel, we need Escribano's Approximation Theorem, cf. [7, Theorem 1.1].

Theorem 3.1. *Let $X \subset R^n$ be a definable open set, let $f : X \to R$ be a definable $\mathscr{C}^m$ function, and let $\varepsilon : X \to (0,\infty)$ be a definable continuous function. Then there is a definable $\mathscr{C}^{m+k}$ function $g : X \to R$ such that for $x \in X$ and $|\alpha| \leq m$*

$$|f_\alpha(x) - g_\alpha(x)| < \varepsilon(x).$$

We now solve the extension problem for functions defined on special sets. A definable function $f : X \to R$ is called a (definable) $\mathscr{C}^m$ function, if there exists a definable open neighborhood U of X and a $\mathscr{C}^m$ function $F : U \to R$ such that $F|_X = f$. For an integer $0 \leq d < n$, we set

$$X_0 := \{(x, 0, \dots, 0) : x \in X\},$$

which is a subset of R^n. We use $\overline{A}$, A° and ∂A to denote the closure, interior and frontier of a set A, respectively.

Lemma 3.2. *Let $d < n$ and $X \subset R^d$ be a definable open set. Let $(f_\alpha)_{|\alpha|\le m} : \overline{X_0} \to R$ be an definable m times Fréchet differentiable function, such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le m$*

(i) *f_α is $\mathscr{C}^m$ smooth in X_0,*
(ii) *$f_\alpha = 0$ on ∂X_0.*

Then, for every definable open neighborhood U of X_0, there exists a definable $\mathscr{P}^m$ function $F : R^n \to R$ such that

(a) *F is $\mathscr{C}^{3m}$ smooth outside of $\overline{X_0}$,*
(b) *$\operatorname{supp}(F) \subset \overline{U}$,*
(c) *$F_\alpha = f_\alpha$ on $\overline{X_0}$ for $|\alpha| \le m$.*

Proof. Step 1: We define the function $h : X \times R^{n-d} \to R$ by

$$h(x, y) := \sum_{\substack{|\alpha|\le m \\ a_1+\cdots+a_d=0}} \frac{f_\alpha(x,0)}{\alpha!} y^\alpha$$

for $x \in X$ and $y \in R^{n-d}$. According to property (i), the function h is definable and m times continuously differentiable. Moreover, for $|\alpha| \le m$,

$$D_\alpha h = f_\alpha \text{ on } X_0.$$

The function h is not necessarily $\mathscr{C}^{3m}$ smooth. Let $\varepsilon : X \times R^{n-d} \to (0, \infty)$ be the definable continuous function given by

$$\varepsilon(x, y) := \frac{y^{2m}}{1 + y^{2m}} \operatorname{dist}(x, \partial X), \quad (x, y) \in X \times R^{n-d}.$$

We apply Theorem 3.1 to h and ε in place of f and ε, and we obtain a definable $\mathscr{C}^{3m}$ function $g : X \times R^{n-d} \backslash X_0 \to R$ such that for all $|\alpha| \le m$ and $(x, y) \in X \times (R^{n-d} \backslash \{0\})$,

$$|g_\alpha(x, y) - h_\alpha(x, y)| < \varepsilon(x, y).$$

Note that for each $\xi \in X_0$

$$\varepsilon(\eta) \text{ is } o(\|\eta - \xi\|^m) \text{ as } \eta \to \xi,$$

so that

$$|h(\eta) - g(\eta)| \text{ is } o(\|\eta - \xi\|^m) \text{ as } \eta \to \xi.$$

Let $G : X \times R^{n-d}$ denote the unique continuous extension of g to $X \times R^{n-d}$, which is definable. Then G is m times Peano differentiable, and, by the choice of h and ε,

$$G_\alpha = f_\alpha \text{ on } X_0$$

for all multi-indices α with $|\alpha| \le m$.

Step 2: Let $\phi : X \to (0,\infty)$ be a definable $\mathscr{C}^{3m}$ function, which satisfies for all $x \in X$

$$\phi(x) < \min\{1, \operatorname{dist}((x,0), \partial U)\},$$
$$\phi(x) < \frac{\operatorname{dist}(x, \partial X)^{m+1}}{1 + \|x\| + \max\{|G_\alpha(x)| : |\alpha| \le m\}}.$$

Furthermore, let $\rho : R \to [0,1]$ be a definable $\mathscr{C}^{3m}$ function that satisfies $\rho(t) = 1$ for $|t| \le 1/2$, and $\rho(t) = 0$ for $|t| \ge 1$.

We define the function $F : R^n \to R$ by

$$F(x,y) := \begin{cases} \rho\big(\frac{y}{\phi(x)}\big) G(x,y), & \text{if } x \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: The support of F is evidently contained in $\overline{U}$. The function F is $\mathscr{C}^{3m}$ smooth in $X \times R^{n-d} \setminus X_0$ as well as in

$$R^n \setminus \overline{\{(x,y) : x \in X, y \in R^{n-d}, \|y\| < \phi(x)\}},$$

so that F is $\mathscr{C}^{3m}$ smooth in $R^n \setminus \overline{X_0}$. Moreover, the function F is m times Peano differentiable in $R^n \setminus \partial X_0$. It remains to prove the Peano differentiability at every point of ∂X_0.

Let $(x,y) \in X \times R^{n-d}$ with $\|y\| < \phi(x)$. Then

$$\begin{aligned} |F(x,y)| \le |G(x,y)| &\le \sum_{\substack{|\alpha| \le m \\ a_1 + \cdots + a_d = 0}} \left| \frac{f_\alpha(x,0)}{\alpha!} y^\alpha \right| + \varepsilon(x,y) \\ &\le n^m \operatorname{dist}(x, \partial X)^{m+1} + \operatorname{dist}(x, \partial X)^{m+1}. \end{aligned}$$

Hence, $F(\eta)$ is $o(\|\eta - \xi\|^m)$ as $\eta \to \xi$ for every $\xi \in \partial X_0$.

Finally, the functions F_α and f_α coincide on $\overline{X_0}$ for $|\alpha| \le m$. □

Next, we generalize the previous lemma to functions defined on sets which are graphs of definable Lipschitz continuous $\mathscr{C}^m$ functions. This requires the notion of Peano differentiable functions whose domain is R^k for some k. By $\pi_i : R^k \to R$ we denote the projection onto the i-th coordinate. For $|\alpha| \le m$ let $f_\alpha : X \to R^k$ be a definable function. We say that $(f_\alpha)_{|\alpha| \le m}$ is m times Peano differentiable if $(\pi_i \circ f_\alpha)_{|\alpha| \le m}$ is m times Peano differentiable for every $i = 1, \dots, k$. We further set

$$p_{f,x}(y) := \sum_{1 \le |\alpha| \le m} \frac{f_\alpha(x)}{\alpha!} (y - x)^\alpha.$$

Remark 3.3. Let $(f_\alpha)_{|\alpha| \le m} : X \to Y$ and $(g_\alpha)_{|\alpha| \le m} : Y \to Z$ be definable m times Peano differentiable functions. Then their composition $(h_\alpha)_{|\alpha| \le m} : X \to Z$ is also m times Peano differentiable. In particular, $h_{(0,\dots,0)} := g_{(0,\dots,0)} \circ f_{(0,\dots,0)}$ and each $h_\alpha(x)$ is given as the α-th coefficient of the polynomial $p_{g,f(x)} \circ p_{f,x}(y)$ divided by $\alpha!$. Note, that the functions h_α are polynomials in the variables $g_\beta \circ f$ and f_β where $|\beta| \leq m$. If X or Y are not open sets, this definition of the h_α is chosen to be compatible with the case of open sets.

The generalization of Lemma 3.2 reads as follows.

Lemma 3.4. *Let $X \subset R^d$ be a definable open set, and let $h : X \to R^{n-d}$ be a definable Lipschitz continuous $\mathscr{C}^{2m}$ function. Set $Y := \Gamma(h)$. Let $(g_\alpha)_{|\alpha| \leq m} : \overline{Y} \to R$ be a definable m times Fréchet differentiable function, such that for $|\alpha| \leq m$*

(i) *g_α is $\mathscr{C}^m$ smooth in Y,*
(ii) *$g_\alpha = 0$ on ∂Y.*

Then for every definable open neighborhood V of Y, there is a definable m times Peano differentiable function $G : R^n \to R$ such that

(a) *G is $\mathscr{C}^{2m}$ smooth outside of $\overline{Y}$,*
(b) *$\operatorname{supp}(G) \subset \overline{V}$,*
(c) *$G_\alpha = g_\alpha$ on $\overline{Y}$ for all $|\alpha| \leq m$.*

Proof. Step 1: Let $\psi : \overline{X} \times R^{n-d} \to \overline{X} \times R^{n-d}$ be the function defined by

$$\psi(x, y) := (x, y + h(x)).$$

Then the function ψ is $\mathscr{C}^{2m}$ smooth in $X \times R^{n-d}$, and it is Lipschitz continuous with Lipschitz continuous inverse on $\overline{X} \times R^{n-d}$. Let L be a Lipschitz constant that is valid for both functions ψ and ψ^{-1}.

Step 2: We define the functions $(f_\alpha)_{|\alpha| \leq m}$ as follows. On X_0, we define the functions f_α according to Remark 3.3 with f, g and ψ in place of h, g and f. Therefore, the f_α are $\mathscr{C}^m$ functions on X_0. For any $\eta \in \partial Y$, property (ii) implies that

$$g(\zeta) \text{ is } o(\|\zeta - \eta\|^m) \text{ as } \zeta \to \eta.$$

Hence, for every $\xi \in \partial X_0$,

$$f(\zeta) = g(\psi(\zeta)) \text{ is } o(\|\psi(\zeta) - \psi(\xi)\|^m) \text{ as } \zeta \to \xi,$$

so that the Lipschitz continuity of ψ implies that

$$f(\zeta) \text{ is } o(\|\zeta - \xi\|^m) \text{ as } \zeta \to \xi.$$

Thus, if we set $f_\alpha(\xi) := 0$ for $\xi \in \partial X_0$, the function $f_{(0,\ldots,0)}$ together with the functions f_α, where $1 \leq |\alpha| \leq m$, is m times Peano differentiable and satisfies the conditions of Lemma 3.2.

Step 3: Let U be the intersection of the sets

$$\{x : \operatorname{dist}(x, X_0) < \operatorname{dist}(x, \partial X_0)\} \;\text{ and }\; \psi^{-1}(V).$$

Then, by Lemma 3.2, there is a definable $\mathscr{P}^m$ function $F : R^n \to R$ that satisfies the conclusions of Lemma 3.2. We claim that $G : R^n \to R$,

$$G(u) := \begin{cases} F(\psi^{-1}(u)), & \text{if } u \in \psi(U), \\ 0, & \text{otherwise,} \end{cases}$$

suits the desired properties.

Property (a) is evidently satisfied, and G is $\mathscr{C}^{2m}$ smooth outside of $\overline{\psi(U)}$. Moreover, the function F is $\mathscr{P}^m$ smooth in $R^n \setminus \partial Y$, and, by construction,

$$F_\alpha = f_\alpha \text{ in } Y \text{ for } |\alpha| \le m.$$

In order to verify that for every $\eta \in \partial Y$

$$F(\zeta) \text{ is } o(\|\zeta - \eta\|^m) \text{ as } \zeta \to \eta, \tag{3.1}$$

we note that for any $\xi \in \partial X_0$,

$$G(\zeta) \text{ is } o(\|\zeta - \xi\|^m) \text{ as } \zeta \to \xi.$$

So

$$F(\zeta) = G(\psi^{-1}(\zeta)) \text{ is } o(\left\|\psi^{-1}(\zeta) - \psi^{-1}(\eta)\right\|^m) \text{ as } \zeta \to \eta,$$

and hence, by the Lipschitz continuity of ψ^{-1}, equation (3.1) is evident. □

3.2. We prove Theorem 1.2. This requires a special kind of partition of definable sets. In the following theorem we summarize several decomposition concepts in o-minimal geometry, cf. [21, p. 115f] and [12, Theorem 1.4].

Theorem 3.5. *Let $k > 0$ be an integer. Let $A := A_1 \cup \cdots \cup A_r \subset R^n$ be a union of definable sets, and let $f : A \to R^k$ be definable. There exists a finite partition of A into definable $\mathscr{C}^m$ sub-manifolds $B_1, \dots, B_s$, called strata, such that*

(a) *each A_i is the union of some of the strata,*
(b) *each stratum B_j is either open, or, after some linear orthogonal change of coordinates, $B_j = \Gamma(h_j)$, where h_j is a Lipschitz continuous $\mathscr{C}^k$ function with open domain, $j = 1, \dots, s$,*
(c) *$f|B_j$ is a definable $\mathscr{C}^k$ function for all $j = 1, \dots, s$,*
(d) *every B_j has a definable open neighborhood V_j which is disjoint to B_ℓ for $\ell \ne j$ and $\dim(B_\ell) \le \dim(B_j)$.*

In case of item (a), we also say that the partition is *compatible* with $A_1, \dots, A_r$.

The *dimension* of a definable set X is the maximal integer $\dim(X)$ such that X contains a set, which is definably homeomorphic to $R^{\dim(X)}$. According to [21, p. 67], every definable set X satisfies the following inequality:

$$\dim(\partial X) < \dim(X).$$

This implies that $\overline{X}$ has the same dimension as X for every definable set X.

Proof of Theorem 1.2. According to Theorem 3.5, we may select a finite partition of A into definable sets $B_1, \ldots, B_s$ of the form (b) of Theorem 3.5 with $k = 2m$, which is compatible with the sets $A_1, \ldots, A_r$, such that the function f_α restricted to B_i is $\mathscr{C}^m$ smooth for each $|\alpha| \leq m$ and $i = 1, \ldots, s$. By permuting the indices, we may further assume that

$$\dim(B_i) \leq \dim(B_{i+1})$$

for $i = 1, \ldots, s-1$. Hence, for any $t = 1, \ldots, s$, the set $\bigcup_{i=1}^{t} B_i$ is closed.

We prove by induction on t that there exists a definable $\mathscr{P}^m$ function $F_t : R^n \to R$, such that $F_\alpha = f_\alpha$ on $\bigcup_{i=1}^{t} B_i$ for all $|\alpha| \leq m$, and such that F is $\mathscr{C}^{2m}$ smooth outside of $\bigcup_{i=1}^{t} B_i$.

The case $t = 1$ is evident by Lemma 3.4.

Suppose that we have constructed F_t. For each $|\alpha| \leq m$ let $g_\alpha : \overline{B_{t+1}} \to R$ be defined by $g_\alpha := f_\alpha - (F_t)_\alpha$. Then each g_α vanishes on ∂B_{t+1}, and g_α restricted to B_{t+1} is a $\mathscr{C}^m$ function as F_t is $\mathscr{C}^{2m}$ smooth outside of $\bigcup_{i=1}^{t} B_i$.

Select, by Theorem 3.5 (d), a definable open neighborhood V of B_{t+1}, which has empty intersection with $\bigcup_{i=1}^{t} B_i$. Then, by Lemma 3.4, there exists a definable $\mathscr{P}^m$ function $G : R^n \to R$, which satisfies the conclusions of Lemma 3.4. Therefore, $F_{t+1} := G + F_t$ satisfies the desired properties.

Set $F := F_s$. □

3.3 Corollaries and remark. As an immediate consequence of Theorem 1.2 we note the o-minimal version of Theorem 1.1.

Corollary 3.6. *Let $A \subset R^n$ be closed and definable, and let $(f_\alpha)_{|\alpha| \leq m} : A \to R$ be a definable $\mathscr{F}^m$ function relative to A. Then there is a definable $\mathscr{P}^m$ function $F : R^n \to R$, such that*

$$F_\alpha = f_\alpha \text{ on } A \text{ for all } |\alpha| \leq m.$$

Corollary 3.7. *Let $A \subset R$ be a definable closed set, and let $(f_\alpha)_{|\alpha| \leq m} : A \to R$ be definably m times Peano differentiable relative to A. Then there is a definable m $\mathscr{P}^m$ function $F : R \to R$ such that $F_\alpha = f_\alpha$ on A for $|\alpha| \leq m$.*

Proof. By o-minimality, the set A is a finite disjoint union of open intervals and singletons. On an open set, the Peano derivatives are determined by the function itself, and by inequality (2.1), a unary definable Peano differentiable function is continuously differentiable. Thus condition $(*)$ of Theorem 1.2 is satisfied. □

Remark 3.8. The condition $(*)$ of Theorem 1.2 is also necessary if $n = 2$.

Proof. This follows from the fact that definable m times Peano differentiable functions defined on an open subset of R^2 are m times continuously differentiable outside of a finite subset. Moreover, every function restricted to a singleton is m times Fréchet differentiable. □

4 Gluing properties

4.1 Open sets. If A is an open set and f and g are m times Peano differentiable in A, then f and g are $\mathscr{P}^m$ equal in A if and only if $g = f$ in A.

Remark 4.1. The gluing property of finitely many definable m times Peano differentiable functions, whose domains are open, reads as follows.

Let $U_1, \dots, U_r$ be definable open subsets of R^n, and for $i = 1, \dots, r$ let $f_i : U_i \to R$ be definable m times Peano differentiable functions such that $f_i = f_j$ on $U_i \cap U_j$ for $1 \leq i, j \leq r$. Then there is a definable m times Peano differentiable function $F : \bigcup_i U_i \to R$ such that $F = f_i$ on U_i for $i = 1, \dots, r$.

The proof is a standard application of definable $\mathscr{C}^m$ partition of unity, and we omit it. If $R = \mathbb{R}$, then definability is not needed, and $r = \infty$ is also allowed.

4.2 Closed sets. More difficulties appear if we want to glue definable weakly m times Peano differentiable functions with closed domains. We do not know whether there exist corresponding studies for gluing m times Peano differentiable functions without the o-minimality assumption on the sets.

Lemma 4.2. *Let $C \subset R^n$ be a set of the form of Theorem* 3.5 (b)*, and let U be a definable open neighborhood of C. Let $f : R^n \to R$ be an m times Peano differentiable function, which is $\mathscr{P}^m$ equal to the zero function in ∂C. Then there is an m times Peano differentiable function $F : R^n \to R$, which vanishes outside of U such that F and f are $\mathscr{P}^m$ equal in $\overline{C}$. If f is definable, we may choose F to be definable.*

Proof. If C is open, then set $F = f$ in $\overline{C}$ and $F = 0$ outside of $\overline{C}$. Otherwise, after some suitable change of coordinates, we may assume that $C = \Gamma(h)$ where $X \subset R^d$ is an open definable set and $h : X \to R^{n-d}$ is a definable Lipschitz continuous $\mathscr{C}^m$ function. Let $\varphi : R^d \to [0, \infty)$ be a definable $\mathscr{C}^m$ function that vanishes outside of X, and which satisfies

$$0 < \varphi(x) < \operatorname{dist}(h(x), R^n \setminus U)$$

for every $x \in X$. Let $\rho : R \to [0, 1]$ be a definable $\mathscr{C}^m$ function which equals 1 in $[-1/2, 1/2]$ and vanishes outside of $(-1, 1)$. Then the function $F : R^n \to R$,

$$F(x, y) := \begin{cases} f(x, h(x))\rho\big(\frac{y-h(x)}{\varphi(x)}\big), & \text{if } x \in X, \\ 0, & \text{otherwise,} \end{cases}$$

suits the desired properties. Moreover, as h, ρ and φ are definable, the function F is definable if f is definable. □

Proof of Theorem 1.3. According to Theorem 3.5, we select a finite partition of R^n that is compatible with the sets $A_1, \dots, A_r$, and we denote by $B_1, \dots, B_k$ the strata which are contained in at least one of the sets A_s. Without loss of generality we may assume that the strata are ordered in such a way that

$$\dim(B_i) \leq \dim(B_{i+1})$$

for $i = 1, \dots, k-1$. Note that the set $\bigcup_{i=1}^{\ell} B_i$ is closed, and that there is a definable open neighborhood $U_{\ell+1}$ of $B_{\ell+1}$ such that for each $\ell = 1, \dots, k-1$,

$$\bigcup_{i=1}^{\ell} B_i \cap U_{\ell+1} = \emptyset.$$

We prove by induction on ℓ that there exists a $\mathscr{P}^m$ function $F_\ell : R^n \to R$ such that for every $i = 1, \dots, k$ and $s = 1, \dots, r$ the functions F_ℓ and f_i are $\mathscr{P}^m$ equal in $A_s \cap \bigcup_{j=1}^{\ell} B_j$.

The case $\ell = 1$ is evident.

Step from ℓ to $\ell+1$: For $i = 1, \dots, k$ let $h_i = F_\ell - f_i$. Then h_i and the zero function are $\mathscr{P}^m$ equal in $\bigcup_{i=1}^{\ell} B_i$. Note that $\partial B_{\ell+1}$ is contained in $\bigcup_{i=1}^{\ell} B_i$. By Lemma 4.2 there is a $\mathscr{P}^m$ function $g_\ell : R^n \to R$ which is $\mathscr{P}^m$ equal to h_i in $A_s \cap \overline{B_{\ell+1}}$, $s = 1, \dots, r$, and which vanishes outside of $U_{\ell+1}$. Now $F_{\ell+1} = F_\ell - g_\ell$ satisfies the desired properties.

Set $F = F_k$. If the functions $f_1, \dots, f_r$ are additionally definable, then, by Lemma 4.2, we may select the functions F_i, h_i and g_i to be definable, so that F is definable. $\square$

5 Weakly m times Peano differentiable functions

Now we consider another extension problem. First we make weak Peano differentiability precise.

Definition 5.1. A function $f : A \to R$ is called *weakly m times Peano differentiable* if there is for every $a \in A$ a polynomial $p \in R[X]$ with $\deg(p) \le m$ and $p(0) = 0$ such that

$$f(x) - f(a) = p(x-a) + o(\|x-a\|^m) \text{ as } x \to a.$$

Note that we omit the word *weak* if the set A is open.

5.1 One-dimensional sets. Our first aim to prove extendability for weakly m times Peano differentiable functions defined on definable closed sets of dimension 1. In this case we can treat both definable and arbitrary functions.

Lemma 5.2. *Let $h : (b,c) \to R^{n-1}$ be a definable Lipschitz continuous $\mathscr{C}^m$ mapping, and let U be a definable open neighborhood of $\Gamma(h)$. If $f : \overline{\Gamma(h)} \to R$ is a weakly $\mathscr{P}^m$ function such that for $a \in \partial\Gamma(h)$,*

$$f(x) \text{ is } o(\|x-a\|^m) \text{ as } x \to a,$$

then there is a $\mathscr{P}^m$ function $F : R^n \to R$ which satisfies

(a) $F = f$ *on* $\overline{\Gamma(h)}$,
(b) $\operatorname{supp}(F) \subset \overline{U}$.

If f is definable we may choose F to be definable.

Proof. The function h is Lipschitz continuous and definable, so that h extends to $[a,b]$ as Lipschitz continuous function $\overline{h}$ with Lipschitz constant $L > 0$. Therefore,

$$f(t,h(t)) \text{ is } o(|t-a|^m) \text{ as } t \to a$$

for every $a \in \partial(b,c)$. In addition, the map $t \mapsto f(t,h(t))$ is a weakly $\mathscr{P}^m$ function on (b,c), so that $t \mapsto f(t,\overline{h(t)})$ is weakly $\mathscr{P}^m$ in $[a,b]$. Let $\varphi : (b,c) \to (0,\infty)$ be a definable $\mathscr{C}^m$ function such that the set

$$V := \{x = (x_1,\ldots,x_n) : \|x - (x_1,h(x_1))\| < \varphi(x_1)\}$$

is contained in U, and let $\rho : R \to [0,1]$ be a definable $\mathscr{C}^m$ function which equals 1 in $[-1/2,1/2]$ and vanishes outside of $(-1,1)$. Then, the function $F : R^n \to R$,

$$F(x_1,\ldots,x_n) := \begin{cases} f(x_1,h(x_1))\rho\left(\frac{\|(x_2,\ldots,x_n)-h(h_1)\|}{\varphi(x_1)}\right), & \text{if } x_1 \in (b,c), \\ 0, & \text{otherwise,} \end{cases}$$

suits our needs. Note that the functions h, φ and ρ are definable, so that F is definable if f is definable. □

The following proposition answers affirmatively the extension property for weakly $\mathscr{P}^m$ functions defined on closed definable subsets of dimension 1.

Proposition 5.3. *Let $A \subset R^n$ be a definable closed set of dimension 1. Then, for every weakly m times Peano differentiable function $f : A \to R$ there exists a $\mathscr{P}^m$ function $F : R^n \to R$ such that*

$$F = f \text{ on } A.$$

If f is definable, then F can be chosen to be definable.

Proof. We select a finite partition $B_1,\ldots,B_s$ of A into sets with properties (a), (b), and (d) of Theorem 3.5. Let $B_1,\ldots,B_r$ denote the singletons, and $B_{r+1},\ldots,B_s$ be the sets of dimension 1.

For $i = 1,\ldots,r$, let the polynomial p_i be chosen in such a way that

$$f(x) - f(b_i) = p_i(x-b_i) + o(\|x-b_i\|^m) \text{ as } x \to b_i$$

where $B_i = \{b_i\}$.

Then the function $G : R^n \to R$ which is defined by

$$G(x) = \sum_{i=1}^{r} \rho\Big(\frac{x-b_i}{r}\Big)(f(b_i) + p_i(x-b_i))$$

is m times Peano differentiable and definable. For each $j = r+1,\ldots,s$, the function $(f-G)$ restricted to B_j additionally satisfies, after some suitable change of coordinates, the conditions of Lemma 5.2.

We select for each $j = r+1, \dots, s$ a definable open neighborhood U_j of B_j which has empty intersection with B_ℓ for $\ell \neq j$. Lemma 5.2 provides $\mathscr{P}^m$ functions $g_j : R^n \to R$ for $j = r+1, \dots, s$, such that $\operatorname{supp}(g_j) \subset \overline{U_j}$ and $g_j(b) = (f - G)(b)$, $b \in \overline{B_j}$.

So,

$$F = G + \sum_{j=r+1}^{s} g_j$$

suits the desired properties.

If f is definable, then the functions g_j can be chosen definable by Lemma 5.2, so that F is definable. $\square$

5.2 Peano differentiable manifolds. The definition of Peano differentiable manifolds is similar to that of $\mathscr{C}^m$ manifolds. Here, all manifolds are embedded. A definable set $S \subset R^n$ is called a definable $\mathscr{P}^m$ manifold if for every $x \in S$ there exists a definable open neighborhood U of x and a definable $\mathscr{P}^m$ diffeomorphism Φ from U to the unit ball $B_1(0)$ such that $\Phi(x) = 0$ and $\Phi(U \cap S) = \{(x_1, \dots, x_n) \in B_1(0) : x_{n-d+1} = \cdots = x_n = 0\}$. An atlas of a definable $\mathscr{P}^m$ manifold is called *definable* if all charts are definable.

By [2, 2.4], every definable $\mathscr{C}^m$ manifold has a finite definable special atlas (of $\mathscr{C}^m$ charts); that is, a finite definable atlas whose charts are linear projections. This is a consequence of the continuity of the tangent mapping of a $\mathscr{C}^m$ manifold.

The tangent mapping of $\mathscr{P}^m$ manifolds is in general not continuous, and we do not know whether or not every definable $\mathscr{P}^m$ manifold possesses a finite definable atlas. Thus we restrict ourselves to manifolds with finite definable special atlases.

Proposition 5.4. *Let $A \subset R^n$ be a closed definable $\mathscr{P}^m$ manifold, which possesses a finite definable special atlas. Let $f : A \to R$ be a weakly $\mathscr{P}^m$ function. Then there is a $\mathscr{P}^m$ function $F : R^n \to R$ such that $F = f$ on A. If F is definable, we can choose F to be definable.*

Proof. Let $\phi_i : S_i \to U_i \subset R^{d_i}$ be the charts of the atlas, $i = 1, \dots, r$. Then each ϕ_i extends to a definable $\mathscr{P}^m$ diffeomorphism Φ_i from a an open definable neighborhood of S_i in R^n to an open definable neighborhood W_i of $U_i \times \{0\}$ in $U_i \times R^{n-d_i}$. We may additionally assume that if $(u, y) \in W_i$, then the segment connecting $(u, 0)$ and (u, y) is contained in W_i. For each i, we extend $f \circ \phi_i^{-1}$ to the $\mathscr{P}^m$ function F_i defined on W_i by setting $F_i(u, y) = f_i \circ \phi_i^{-1}(u)$. The sets $U_1, \dots, U_r$ cover A. Let $U_0 := R^n \setminus A$. Select a definable $\mathscr{C}^m$ partition of unity $\varphi_0, \dots, \varphi_r : R^n \to R$ subordinate to the sets $U_0, \dots, U_r$. Then the function $F : R^n \to R$ given by

$$F := \sum_{i=1}^{r} \varphi_i F_i \circ \phi_i$$

has the desired properties. As the functions $\varphi_1, \dots, \varphi_r$ and $\phi_1, \dots, \phi_r$ are definable, the function F is definable if f is definable. $\square$

5.3 Sets in R^2. We prepare the proof of Theorem 1.4 by the following lemma.

Lemma 5.5. *Let $f, g : (a,b) \to R$ be definable $\mathscr{C}^m$ functions with $f(t) < g(t)$ for all $t \in (a,b)$. Let*

$$C := \{(x,y) : x \in (a,b) \wedge f(x) < y \leq g(x)\}.$$

Let $F : C \to R$ be an m times weakly Peano differentiable function. Then there exists a definable open neighborhood U of C and an m times Peano differentiable function $G : U \to R$ such that $G = F$ on C.

Proof. By applying the function $\psi : (a,b) \times R \to (a,b) \times R$ defined by

$$\psi(x,y) := (x, y - g(x)),$$

we may assume that $g \equiv 0$. Consider the linear system

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & -2 & \dots & -m \\ 0 & (-1)^2 & (-2)^2 & \dots & (-m)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & (-1)^m & (-2)^m & \dots & (-m)^m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{5.1}$$

The above $(m+1)\times(m+1)$-matrix is of Vandermonde type, whose generators $0, -1, \dots, -m$ are pairwise distinct. Thus this matrix is invertible, and the system (5.1) has a unique solution $a_0, \dots, a_m$. We set

$$U := \left\{(x,y) : x \in (a,b) \wedge f(x) < y < \frac{-f(x)}{m+1}\right\},$$

and define $G : U \to R$ by

$$G(x,y) := \begin{cases} F(x,y), & y < 0, \\ \sum_{k=0}^m a_k F(x, -ky), & y \geq 0. \end{cases} \tag{5.2}$$

By the choice of $a_0, \dots, a_n$, the functions F and $\sum_{k=0}^m a_k F(x, -ky)$ are $\mathscr{P}^m$ equal in $(a,b) \times \{0\}$, so that they glue together to the $\mathscr{P}^m$ function G.

Note that if F is definable, then also G is definable. □

Corollary 5.6. *Let f, g, F, C and U be as in Lemma 5.5. If F is $o(\|x - b\|^m)$ as $x \to b$ for every $b \in \partial\Gamma(g)$, then we may assume that the function G is $\mathscr{P}^m$ equal to 0 in $\Gamma(-f/(m+2))$.*

Proof. Let $\rho : R \to [0,1]$ be a definable $\mathscr{C}^m$ function that equals 1 for $t \leq 0$ and vanishes for $t \geq 1/2$. Then the function G from equation (5.2) multiplied with

$$\rho\Big(\frac{y(m+2)}{f(x)}\Big)$$

has the desired property. □

We are now able to prove Theorem 1.4.

Proof. Consider the sets

$$\overline{A^\circ} \text{ and } \overline{A \setminus \overline{A^\circ}}.$$

The dimension of the latter set is bounded by 1, and $\overline{A^\circ} \cap \overline{A \setminus \overline{A^\circ}}$ is a finite set. By arguing as in the proof of Proposition 5.3 we may assume that

$$f(x) \text{ is } o(\|x-b\|^m) \text{ as } x \to b$$

for every $b \in \overline{A^\circ} \cap \overline{A \setminus \overline{A^\circ}}$. Hence, by Theorem 1.3 and Proposition 5.3 it remains to prove that f restricted to $\overline{A^\circ}$ extends to R^2, as the extendability of f restricted to $\overline{A \setminus \overline{A^\circ}}$ was already proved in Proposition 5.3. Select a finite partition of $\overline{A^\circ}$ into definable $\mathscr{C}^m$ submanifolds $B_1, \ldots, B_s$. We may additionally assume that $\dim(B_i) \leq \dim(B_{i+1})$ for $i = 1, \ldots, s-1$. If $B_j = \{b_j\}$ is a singleton contained in the boundary of $\overline{A^\circ}$, there is a polynomial p_j such that f and p_j are $\mathscr{P}^m$ equal in B_j. Select a definable $\mathscr{C}^m$ function $\varphi_j : R^2 \to R$ which equals 1 in a sufficiently small open neighborhood of b_j, and that vanishes in a definable open neighborhood of the other B_ℓ with $\dim(B_\ell) = 0$. Then

$$f(x) - \varphi p_j(x - b_j) \text{ is } o(\|x-b_j\|^m) \text{ as } x \to b_j.$$

So we may assume that

$$f(x) \text{ is } o(\|x-b_j\|^m) \text{ as } x \to b_j$$

for each j with $\dim(B_j) = 0$. If B_j is contained in the boundary of $\overline{A^\circ}$ such that $\dim(B_j) = 1$, then there is a definable open neighborhood U_j of B_j and a $\mathscr{P}^m$ function $G_j : U_j \to R$ such that G_j and f are $\mathscr{P}^m$ in $U \cap A$. Moreover, the function f is $\mathscr{P}^m$ equal to 0 at every point $b \in \overline{B_j} \setminus B_j$. We glue the G_j and $f|\overline{A^\circ}$ together to the function F. By Corollary 5.6, we can extend this function to R^2 by setting $F(x) = 0$ outside of a closed definable neighbourhood of $\overline{A^\circ}$.

Note that if f is definable then the functions G_j are definable, and so F is definable. □

6 Open questions

Consider the Example 2.4 for $R = \mathbb{R}$. We do not know whether f can be extended as (non-definable) 2 times Peano differentiable function or not. It would be interesting whether or not a definable $\mathscr{P}^m$ function $f : A \to \mathbb{R}$ is the restriction of a definable $\mathscr{P}^m$ function $F : \mathbb{R}^n \to \mathbb{R}$ if and only if f is the restriction of a (non-definable) $\mathscr{P}^m$ function $G : \mathbb{R}^n \to \mathbb{R}$.

Furthermore, the condition $(*)$ of Theorem 1.2 is only known to be necessary in the case $n = 2$. So far, no definable m times Peano differentiable function with open domain is known that does not satisfy this condition. The following question arises: Can condition $(*)$ be weakened, or does every definable $\mathscr{P}^m$ function with open domain satisfy $(*)$?

Finally, can Theorem 1.4 be generalized to higher dimension, or what are the sets that admit extendability?

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